PHYSICAL REVIEW E VOLUME 47, NUMBER 4 APRIL 1993

Reformulation of recursive-renormalization-group-based subgrid modeling of turbulence

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(Received 27 August 1992; revised manuscript received 7 December 1992)

An alternative development of the recursive-renormalization-group (RNG) theory for the subgrid modeling of turbulence is presented which is now independent of the order in which the subgrid averaging is performed. The relevant approximations, perturbation ordering, and the averaging process are explicitly considered. In particular, it is shown that, of the higher-order nonlinearities introduced into the RNG Navier-Stokes equation, only the third-order nonlinearity appears at the desired level of the perturbation expansion. Moreover, these triple-velocity product terms appear at the same order as that of the eddy viscosity which is generated by the RNG subgrid-elimination procedure. These third-order nonlinearities also play a major role in the energy-balance equation with the corresponding energy-transfer process resulting in an analytic eddy-viscosity formulation which is in agreement with that from closure theories and the results of direct numerical simulations (DNS). This is also confirmed further here by a direct analysis of both large-eddy-simulation and DNS databases for the fluid velocity. Moreover, it is shown that these RNG-induced triple nonlinearities give rise to a backscatter in the energy from small scales to large spatial scales, in agreement with recent closure theories and numerical simulations.

PACS number(s): 47.10.+g, 47.27.Gs

I. INTRODUCTION

Renormalization-group (RNG) procedures, based on the successive elimination of groups of statistically symmetric modes, were first introduced in the study of critical phenomena [1–3] and then to a variety of problems in other areas of physics [3,4]. Invariably these procedures, which reduce the number of relevant degrees of freedom, introduce higher-order nonlinearities into the decimated system. This is to be expected if the original system and the reduced system are to give the same physics. Indeed, in the application of RNG to the two-dimensional (2D) Ising model Wilson [1] found that after the first decimation the spin Hamiltonian no longer just involved the nearest-neighbor interaction but also a diagonal nearest neighbor as well as a four-spin interaction. Since the Ising model is in thermal equilibrium, it was possible to prove that the strength of these higher-level interactions is successively weaker. Wilson [1] then dropped the four-spin interaction, but retained the RNG-induced diagonal-nearest-neighbor interaction. Excellent agreement with the exact solution of the 2D Ising model was then achieved only if this RNG-induced interaction was retained.

In the application of RNG to fluid turbulence, one is now faced with a system far away from equilibrium. In the ϵ-RNG work of Forster, Nelson, and Stephen [5], Fournier and Frisch [6], and Yakhot and Orszag [7] it is customary to introduce a white-noise, zero-mean forcing term into the Navier-Stokes momentum equation. This forcing term is determined by its correlation function. A small parameter ϵ is then introduced into the exponent of this correlation function and exploited in the subsequent perturbation expansion. Provided $\epsilon \ll 1$, it [7] has been shown that all the higher-order nonlinearities introduced by the RNG procedure are irrelevant interactions and can be ignored. However, to recover the Kolmogorov inertial range spectrum one is forced into the limit $\epsilon \to 4$. Unfortunately, in this limit of $\epsilon \to 4$ it can no longer be deduced that the RNG-induced higher-order nonlinearities are ignorable. This is an acknowledged [7] drawback of the ϵ-RNG theories. Kraichnan [8] has concluded that without higher-order nonlinearities the ϵ RNG implies a distant-interaction approximation.

Independently, we have applied a recursive-RNG procedure (originally introduced by Rose [9] to the linear problem of turbulent advection of a passive scalar) to both free-decaying [10] and forced [11] Navier-Stokes turbulence. Again, higher-order nonlinearities are introduced into the renormalized momentum equation. Based on the observation that these higher-order interactions are not only a natural by-product [1,2] of RNG in equilibrium problems but are, in fact, essential in obtaining a correct theory, we [10–12] kept only the third-order nonlinearities and examined their effect on the renormalized eddy viscosity in the momentum equation. In these earlier calculations, higher-order nonlinearities were dropped without justification. Here, we shall argue that this was a correct procedure by showing that the fourth-order nonlinearities are higher order in the RNG perturbation expansion than the third-order nonlinearities. Hence one is justified in dropping these fourth-order nonlinearities from the renormalized momentum equation [just as one is justified in dropping [1,2] the quartic coupling from the 2D RNG-Ising model]. On the other hand, we shall show in Sec. II and Appendix A that the third-order nonlinearities occur at the same level in the perturbation expansion as the eddy viscosity generated by the RNG procedure. This again stresses the importance of retaining the leading-order RNG-induced interactions.

Subgrid-scale modeling is necessary in the numerical studies of turbulence because the required spatial and temporal resolution in computer simulation is unattainable [13]. Since all necessary small (subgrid) scales cannot be resolved numerically, one must resort to some analytical modeling (subgrid modeling) in order to calculate their effects on the numerically resolvable (supergrid) large scales (large-eddy simulations) [14]. Classically, *ad hoc* subgrid models have been used based on phenomenological closure arguments [14–16] and adjustable numerical factors [17], with some care needed to avoid internal inconsistencies [18]. An important goal of the RNG approach is to provide a systematic subgrid-scale-elimination scheme.

The essence of recursive RNG is to separate the turbulent field into two mutually exclusive equations for each wave-number band that is to be decimated: one equation describes the evolution of the subgrid rapidly evolving scales while the other equation describes the evolution on the resolvable slowly evolving scales. The subgrid scales are eliminated by appropriately solving the corresponding dynamical equation, and then substituting this result into the evolution equation for the resolvable scales. An ensemble average is then performed over the subgrid scales. As a result, new triple nonlinearity and nonlocal eddy-damping functions are generated in the averaged supergrid equation. This procedure is then repeated until the last subgrid shell is removed. Unlike ϵ-RNG theories [6,7], rescaling transformations—which are the backbone of usual RNG [1–4]—are necessary. In the systematic elimination of the subgrid shells in the recursive RNG, one can resort to an effective Reynolds number and this point is discussed in some detail by Rose [9].

However, an important question that has not yet been clarified is the order in which the ensemble average is to be performed. In earlier calculations [9–11], it was necessary to perform the ensemble average after the subgrid-scale solution is substituted into the dynamical equation for the resolvable scales. Otherwise, the interaction between the subgrid and resolvable scales would be eliminated—and this interaction is crucial in generating the triple nonlinearities. On physical grounds, however, it is to be expected that the final physics should not be dependent on whether ensemble averaging is performed before or after the subgrid-scale substitution. Here this question is reexamined by considering the relevant approximations and perturbation ordering made in both free-decaying and forced Navier-Stokes turbulence. We then present, in Sec. II, a new formulation of recursive RNG that is independent of the order in which the ensemble average is performed.

Although the recursive RNG has been considered [19] a natural analytic representation for large-eddy simulation (LES), one must demonstrate that this technique will provide as accurate a solution as that obtained from standard closure theories [20–22] at large Reynolds numbers and direct numerical simulation (DNS) based on measurements [23–26] at lower Reynolds numbers. In particular, the closure theories [20–22] were the first to demonstrate that the eddy viscosity should exhibit the following wave-number characteristics: (i) as $k \rightarrow 0$, it should asymptotically approach a constant, while (ii) as $k \rightarrow k_c$, it should exhibit a strong cusp. Here k_c is the boundary between the resolvable and subgrid wave numbers. These predictions were supported by subsequent DNS measurements [23–26]. Current RNG theories [5–12] do not reproduce both of these characteristics. In particular, in ϵ RNG [5–7] there is no wave-number structure in the eddy viscosity since these calculations are restricted to the imposition of the limit $k \rightarrow 0$. On the other hand, recursive RNG [10,11] does yield a wave-number-dependent eddy-viscosity structure. It does asymptotically approach a constant as $k \rightarrow 0$, but the renormalized eddy viscosity in the momentum equation exhibits only a mild cusp as $k \rightarrow k_c$, in qualitative agreement with the closure theories [20–22] and DNS measurements [23–26]. Moreover, it was shown that this cusp behavior was due to the presence of the triple nonlinearities induced by the recursive RNG in both free-decaying [10] and forced [11] Navier-Stokes turbulence. However, a major discrepancy between recursive RNG [10,11] and closure theories [20–22] was in the strength of the eddy-viscosity cusp as $k \rightarrow k_c$.

In Sec. III we resolve this discrepancy by examining the RNG energy-transfer equation. In particular, it is shown how the triple nonlinearities in the renormalized momentum equation contribute to the full eddy viscosity in the energy-transfer equation. An analytic expression for this triple nonlinearity-induced drain-eddy viscosity ν_T is obtained which exhibits not only a strong cusp behavior at k_c, in good agreement with standard closure theories [20–22], but also becomes negative for small k. This small-k behavior of ν_T indicates that there is a backscatter of energy from small scales to large spatial scales, as has been seen in recent numerical results [23,25,26]. This may indicate that the renormalized Navier-Stokes equation may be a better LES model than those models based on subgrid-scale closure theories [20–22] since it is formulated directly on the momentum equation. In this way, one will avoid the difficult task of relating a consistent subgrid model (with subgrid drain and backscatter [20,21,27]) from the energy equation back to the momentum equation [27]. It should be noted, moreover, that this identification has only been achieved for a model problem [25,26]—and not for the Navier-Stokes problem itself.

In Sec. IV we investigate LES and DNS databases for the fluid velocity to study directly the effects of local interactions on the energy-transfer equation—in particular the effects of the coupling of resolvable and subgrid scales on the eddy viscosity. It is precisely these terms which are handled effectively in recursive RNG but completely neglected in ϵ RNG.

In Sec. V we summarize the main results of this paper while some supplementary material is presented in the Appendixes.

II. REFORMULATION OF THE RECURSIVE-RNG PROCEDURE

The evolution of the velocity field in incompressible turbulence is given by the Navier-Stokes equation, in Fourier representation,

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha(\mathbf{k},t)=f_\alpha(\mathbf{k},t)+M_{\alpha\beta\gamma}(k)\int d^3j\, u_\beta(\mathbf{j},t)\times u_\gamma(\mathbf{k}-\mathbf{j},t)\ , \quad (1)$$

with

$$k_\alpha u_\alpha(\mathbf{k},t)=0\ . \quad (2)$$

ν_0 is the molecular viscosity, f_α is the forcing term (with $f_\alpha=0$ in free-decaying turbulence), and summation convention on repeated subscripts is understood. The nonlinear coupling coefficient

$$M_{\alpha\beta\gamma}(k)=k_\gamma D_{\alpha\beta}(k)+k_\beta D_{\alpha\gamma}(k)\ , \quad (3)$$

where $D_{\alpha\beta}$ is the projection operator

$$D_{\alpha\beta}(k)=\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\ . \quad (4)$$

In recursive RNG, the subgrid wave-number region $k_c<k<k_0$ is partitioned into N shells, each with thickness h, $0<h<1$:

$$k_c\equiv k_N\equiv h^N k_0<\cdots<k_i\equiv h^i k_0<\cdots<k_1=hk_0<k_0\ , \quad (5)$$

where k_0 is typically chosen to be on the order of the Kolmogorov dissipation wave number [28] and k_c is at the boundary between the subgrid and resolvable scales. It is convenient to introduce the superscript notation $>$ for subgrid quantities and $<$ for resolvable scale quantities. Thus, in the first subgrid shell $k_1<k<k_0$, the evolution of the subgrid field is given by

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^>(\mathbf{k},t)=f_\alpha^>(\mathbf{k},t)+\lambda M_{\alpha\beta\gamma}(k)\int d^3j[u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+2u_\beta^>(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+u_\beta^>(\mathbf{j},t)u_\gamma^>(\mathbf{k}-\mathbf{j},t)]\ , \quad (6)$$

while for the resolvable scales, $k<k_1$,

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^<(\mathbf{k},t)=f_\alpha^<(\mathbf{k},t)+\lambda M_{\alpha\beta\gamma}(k)\int d^3j[u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+2u_\beta^>(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+u_\beta^>(\mathbf{j},t)u_\gamma^>(\mathbf{k}-\mathbf{j},t)]\ . \quad (7)$$

The parameter λ is introduced to organize the perturbation expansion, and as in all RNG theories, λ will eventually be set to unity. One should note that because of the different k regions in Eqs. (6) and (7), the corresponding $\int d^3j$ are different.

A formal solution to Eq. (6) can be obtained by introducing the Green's function [11] for the first subgrid shell

$$G_0(k,t,\tau)=\exp[-\nu_0 k^2(t-\tau)] \quad (8)$$

so that

$$u_\alpha^>(\mathbf{k},t)=\int d\tau\, G_0(k,t,\tau)\left\{f_\alpha^>(\mathbf{k},\tau)+\lambda M_{\alpha\beta\gamma}(k)\int d^3j[u_\beta^<(\mathbf{j},\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},\tau)+2u_\beta^>(\mathbf{j},\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},\tau)+u_\beta^>(\mathbf{j},\tau)u_\gamma^>(\mathbf{k}-\mathbf{j},\tau)]\right\}\ . \quad (9)$$

A. Forced turbulence

For forced turbulence, the subgrid velocity field is expanded in terms of λ,

$$u_\alpha^>(\mathbf{k},t)=u_\alpha^{>0}(\mathbf{k},t)+\lambda u_\alpha^{>1}(\mathbf{k},t)+\cdots \quad (10)$$

so that from Eqs. (8) and (9), the leading-order subgrid field

$$u_\alpha^{>0}(\mathbf{k},t)=\int d\tau\, G_0(k,t,\tau)f_\alpha^>(\mathbf{k},\tau)\ . \quad (11)$$

The random force is specified by its ensemble average properties

$$\langle f_\alpha^<(\mathbf{k},t)\rangle=f_\alpha^<(\mathbf{k},t),\quad \langle f^>(\mathbf{k},t)\rangle=0\ , \quad (12)$$

with

$$\langle f_\alpha^>(\mathbf{k},t) f_\beta^>(\mathbf{k}',t')\rangle = D_0 D_{\alpha\beta}(k) k^{-y} \delta(\mathbf{k}+\mathbf{k}')\delta(t-t') \ . \tag{13}$$

D_0 defines the intensity of the forcing term. In 3D, the choice of forcing exponent $y=3$ will recover the Kolmogorov energy spectrum [7,11,29] while the choice $y=-2$ corresponds to thermal equilibrium [5,13].

From Eqs. (11) and (12),

$$\langle u_\alpha^{>0}(\mathbf{k},t)\rangle = 0, \quad \langle u_\alpha^<(\mathbf{k},t)\rangle = u_\alpha^<(\mathbf{k},t) \ , \tag{14}$$

and

$$\langle u_\alpha^{>0}(\mathbf{k},t) u_\beta^<(\mathbf{k}',t')\rangle = \langle u_\alpha^{>0}(\mathbf{k},t)\rangle u_\beta^<(\mathbf{k}',t') = 0 \tag{15}$$

since the leading-order subgrid and resolvable scale velocity fields are statistically independent.

The subgrid autocorrelation for homogeneous turbulence takes the form [30]

$$\langle u_\alpha^{>0}(\mathbf{k},t) u_\beta^{>0}(\mathbf{k}',t')\rangle = Q_{\alpha\beta}(\mathbf{k},t,t')\delta(\mathbf{k}+\mathbf{k}') \ , \tag{16}$$

where $Q_{\alpha\beta}$ are the corresponding spectra, given explicitly by Eqs. (11) and (13). For isotropic stationary turbulence $Q_{\alpha\beta}$ simplifies to

$$Q_{\alpha\beta}(\mathbf{k},t,t') = D_{\alpha\beta}(k) Q(|\mathbf{k}|, t-t') \ . \tag{17}$$

The subgrid velocity field at order $O(\lambda)$, from Eqs. (9) and (10), satisfies

$$u_\alpha^{>1}(\mathbf{k},t) = M_{\alpha\beta\gamma}(k)\int d^3j\, d\tau\, G_0(k,t,\tau)[u_\beta^<(\mathbf{j},\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},\tau) + 2u_\beta^{>0}(\mathbf{j},\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},\tau) + u_\beta^{>0}(\mathbf{j},\tau)u_\gamma^{>0}(\mathbf{k}-\mathbf{j},\tau)] \ . \tag{18}$$

B. Free-decaying turbulence

In Free-decaying turbulence, the subgrid velocity is now given by

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^>(\mathbf{k},t) - \lambda M_{\alpha\beta\gamma}(k)\int d^3j u_\beta^>(\mathbf{j},t)u_\gamma^>(\mathbf{k}-\mathbf{j},t) = \lambda M_{\alpha\beta\gamma}(k)\int d^3j [2u_\beta^>(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t) + u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)] \ . \tag{19}$$

It is no longer appropriate to expand $u^>$ in powers of λ as in the forced-turbulence case, Eq. (10). Instead, we subdivide the dynamics into a subgrid scale part that contains the "subgrid-subgrid," "subgrid-resolvable," and "resolvable-resolvable" couplings. When the small-scale fields are broadband, one treats the "subgrid-subgrid" coupling as turbulent. The "subgrid-subgrid" coupling is principally local in wave-number space while the other couplings are nonlocal in wave-number space [31,32]. With this in mind, we decompose the subgrid field $u^>$ into

$$u_\alpha^>(\mathbf{k},t) = u_\alpha^{>b}(\mathbf{k},t) + u_\alpha^{>c}(\mathbf{k},t) \ , \tag{20}$$

where $u^{>b}$ corresponds to the base subgrid-scale turbulence which is described by Eq. (19) with the right-hand side set to zero. $u^{>c}$ is the effect of the large-scale field on the base subgrid turbulence [33].

The formal solution to the base subgrid field $u^{>b}$ is

$$u_\alpha^{>b}(\mathbf{k},t) = \lambda\int d^3j\, d\tau\, G_0(k,t,\tau) M_{\alpha\beta\gamma}(k) \times u_\beta^{>b}(\mathbf{j},\tau)u_\gamma^{>b}(\mathbf{k}-\mathbf{j},\tau) \ . \tag{21}$$

Since the base turbulence can be assumed to be statistically isotropic and homogeneous [33]

$$\langle u_\beta^{>b}(\mathbf{j},\tau)u_\gamma^{>b}(\mathbf{k}-\mathbf{j},\tau)\rangle = D_{\beta\gamma}(j)Q(|\mathbf{j}|,t-t')\delta(\mathbf{k}) = 0 \ , \tag{22}$$

for $\mathbf{k}$ in the subgrid region, i.e., $|\mathbf{k}|>0$, so that the $\delta(\mathbf{k})$ factor is zero. Technically speaking, Eq. (22) shows that the average interaction between the two base subgrid fields at the same vertex is zero. Thus, on averaging Eq. (21),

$$\langle u_\alpha^{>b}(\mathbf{k},t)\rangle = 0 \ . \tag{23}$$

The corresponding equation for the correction to the base subgrid velocity field is approximately given by

$$u_\alpha^{>c}(\mathbf{k},t) \approx \lambda\int d^3j\, d\tau\, G_0(k,t,\tau)M_{\alpha\beta\gamma}(k) \times [u_\beta^<(\mathbf{j},\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},\tau) + 2u_\beta^{>b}(\mathbf{j},\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},\tau)] \ . \tag{24}$$

$u^{>b}$ is introduced into the right-hand side of Eq. (24) to indicate that these are the major contribution to the subgrid-resolvable interaction [33]. This can also be viewed as an assumption on restricting ourselves to weak (to lowest-order) inhomogeneous turbulence in the dynamic equation for the subgrid field. This is to be distinguished from the case of forced turbulence where the leading-order term in the subgrid field is given explicitly by Eq. (11). In addition, the free-decaying subgrid autocorrelation is assumed to follow the inertial range power-law structure since $Q(|\mathbf{k}|)$ is related to the energy spectrum [30]

$$Q(|\mathbf{k}|,t-t') = \frac{E(k,t-t')}{4\pi k^2} \ . \tag{25}$$

For forced turbulence, the subgrid autocorrelation is determined from the forcing correlation function [11].

C. Averaging process

If we perform ensemble averaging in the case of forced turbulence, then from Eqs. (13) and (15),

$$\langle u_\alpha^{>1}(\mathbf{k},t)\rangle = M_{\alpha\beta\gamma}(k)\int d^3j\, d\tau\, G_0(k,t,\tau) \times u_\beta^{<}(\mathbf{j},\tau)u_\gamma^{<}(\mathbf{k}-\mathbf{j},\tau) . \qquad (26)$$

The last term in Eq. (18) vanishes since the interaction between the two leading-order subgrid fields through the same vertex—i.e., the average of this term will lead to a $\delta(\mathbf{k})$ term [similar to that arising in the free-decaying turbulence case, Eq. (22)]. This term cannot contribute since $k > k_1$. We now see one of the advantages of the present formulation over our earlier treatment [11]: a closure approximation need not be invoked to remove the triple subgrid term $u^{>}u^{>}u^{>}$. Thus, for forced turbulence,

$$\langle u_\alpha^{>}\rangle = \langle [u_\alpha^{>0} + \lambda u_\alpha^{>1} + \cdots]\rangle = \lambda\langle u_\alpha^{>1}\rangle + \cdots , \quad \neq 0 \qquad (27)$$

where $\langle u^{>1}\rangle$ is determined from Eq. (26).

To complete the analysis, one can calculate the subgrid autocorrelation

$$\langle u_\beta^{>}(\mathbf{j},t)u_\gamma^{>}(\mathbf{k}-\mathbf{j},t)\rangle = \langle u_\beta^{>0}(\mathbf{j},t)u_\gamma^{>0}(\mathbf{k}-\mathbf{j},t)\rangle + 2\lambda\langle u_\beta^{>1}(\mathbf{j},t)u_\gamma^{>0}(\mathbf{k}-\mathbf{j},t)\rangle \qquad (28)$$

to $O(\lambda)$. Again, the first term in Eq. (28) is zero since it will provide a $\delta(\mathbf{k})$ term and $M_{\alpha\beta\gamma}(k)\delta(\mathbf{k})=0$. The leading-order subgrid correlation is thus $O(\lambda)$ and is readily derived from Eq. (18) by multiplying it by $\mathbf{u}^{>0}$ and performing the subgrid average:

$$\begin{aligned}\langle u_\beta^{>1}(\mathbf{j},t)u_\gamma^{>0}(\mathbf{k}-\mathbf{j},t)\rangle = {} & M_{\beta\beta'\gamma'}(\mathbf{j})\int d^3j'd\tau\, G_0(j,t,\tau)u_{\beta'}^{<}(\mathbf{j}',\tau)u_{\gamma'}^{<}(\mathbf{j}-\mathbf{j}',\tau)\langle u_\gamma^{>0}(\mathbf{k}-\mathbf{j},t)\rangle \\ & + 2M_{\beta\beta'\gamma'}(\mathbf{j})\int d^3j'd\tau\, G_0(j,t,\tau)\langle u_{\beta'}^{>0}(\mathbf{j}',\tau)u_\gamma^{>0}(\mathbf{k}-\mathbf{j},t)\rangle u_{\gamma'}^{<}(\mathbf{j}-\mathbf{j}',\tau) \\ & + M_{\beta\beta'\gamma'}(\mathbf{j})\int d^3j'd\tau\, G_0(j,t,\tau)\langle u_{\beta'}^{>0}(\mathbf{j}',\tau)u_\gamma^{>0}(\mathbf{k}-\mathbf{j},t)u_{\gamma'}^{>0}(\mathbf{j}-\mathbf{j}',\tau)\rangle .\end{aligned}$$

The first term is zero since $\langle \mathbf{u}^{>0}\rangle=0$, while the third term is higher order in λ because of the triple subgrid term $u^{>0}u^{>0}u^{>0}$. Hence, to $O(\lambda)$,

$$\langle u_\beta^{>1}(\mathbf{j},t)u_\gamma^{>0}(\mathbf{k}-\mathbf{j},t)\rangle = 2M_{\beta\beta'\gamma'}(j)\int d^3j'd\tau\, G_0(j,t,\tau)\langle u_{\beta'}^{>0}(\mathbf{j}',\tau)u_\gamma^{>0}(\mathbf{k}-\mathbf{j},t)\rangle u_{\gamma'}^{<}(\mathbf{j}-\mathbf{j}',\tau) . \qquad (29)$$

Similarly, for free-decaying turbulence, the subgrid ensemble average on Eq. (20) yields, at $O(\lambda)$,

$$\begin{aligned}\langle u_\alpha^{>}(\mathbf{k},t)\rangle &= \langle u_\alpha^{>b}(\mathbf{k},t)\rangle + \langle u_\alpha^{>c}(\mathbf{k},t)\rangle \\ &= \lambda M_{\alpha\beta\gamma}(k)\int d^3j\, d\tau\, G_0(k,t,\tau)u_\beta^{<}(\mathbf{j},\tau)u_\gamma^{<}(\mathbf{k}-\mathbf{j},\tau) .\end{aligned} \qquad (30)$$

Again, this should be contrasted with the usual assumption [10] $\langle u^{>}\rangle=0$. The subgrid autocorrelation for free decay

$$\begin{aligned}\langle u_\beta^{>}(\mathbf{j},t)u_\gamma^{>}(\mathbf{k}-\mathbf{j},t)\rangle &\approx \langle u_\beta^{>b}(\mathbf{j},t)u_\gamma^{>b}(\mathbf{k}-\mathbf{j},t)\rangle + 2\langle u_\beta^{>c}(\mathbf{j},t)u_\gamma^{>b}(\mathbf{k}-\mathbf{j},t)\rangle \\ &= 2\lambda M_{\beta\beta'\gamma'}(j)\int d^3j'd\tau\, G_0(k,t,\tau)\langle u_{\beta'}^{>b}(\mathbf{j}',\tau)u_\gamma^{>b}(\mathbf{k}-\mathbf{j},\tau)\rangle u_{\gamma'}^{<}(\mathbf{j}-\mathbf{j}',\tau) ,\end{aligned} \qquad (31)$$

where again the first term will not contribute since $M_{\alpha\beta\gamma}(k)\delta(\mathbf{k})=0$ (i.e., the interaction between two base velocity fields at the same vertex does not contribute). We have also dropped all nonlinear terms involving $u^{>c}$. It should be noted that one can readily write down a unified framework with which to handle either forced or free-decay turbulence. Such a unified framework [12] has been found to be useful in examining the effects of helicity on subgrid-scale closure [34].

Equations (26) and (30) can be considered as self-consistency constraints on the present closure scheme. It is precisely these constraints that will allow us to first perform the ensemble averaging over the subgrid scales in the dynamical equation for the resolvable scales—a procedure that was not permitted in earlier RNG formulations [10,11].

D. Resolvable scale momentum equation

We now present an alternative recursive RNG in which ensemble averaging over the subgrid scales is performed first—a method that in the dynamo literature [35] is known as multiple scale smoothing. As in our earlier multiple scale smoothing. As in our earlier multiple scale elimination technique [10,11]—in which the subgrid-scale averaging must be performed as the last operation—we work to $O(\lambda^2)$. This treatment applies to either free-decaying (with $f_\alpha=0$) or forced turbulence.

The resolvable scale fields satisfy the subgrid-averaged Eq. (7):

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^<(\mathbf{k},t)=f_\alpha^<(\mathbf{k},t)+\lambda M_{\alpha\beta\gamma}(k)\int d^3j[u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+2\langle u_\beta^>(\mathbf{j},t)\rangle u_\gamma^<(\mathbf{k},\mathbf{j},t)$$
$$+\langle u_\beta^>(\mathbf{j},t)u_\gamma^>(\mathbf{k}-\mathbf{j},t)\rangle] \,, \tag{32}$$

where the subgrid-averaged terms are given by Eqs. (26)–(31). Thus, after eliminating the first subgrid shell, the Navier-Stokes equation for the resolvable scales is

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha(\mathbf{k},t)+\int_{-\infty}^{t} d\tau\, \eta_0(k,t,\tau)u_\alpha(\mathbf{k},\tau)$$
$$=f_\alpha(\mathbf{k},t)+\lambda M_{\alpha\beta\gamma}(k)\left[\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)+2\lambda\int d^3j\, d^3j' M_{\beta\beta'\gamma'}(j)G_0(j,t,\tau)u_{\beta'}(\mathbf{j}',\tau)u_{\gamma'}(\mathbf{j}-\mathbf{j}',\tau)u_\gamma(\mathbf{k}-\mathbf{j},t)\right] . \tag{33}$$

For notational simplicity, the superscript < is dropped since all the velocity fields are in the resolvable wave-number region, and the eddy damping function η_0 is given by

$$\eta_0(k,t,\tau)=-4M_{\alpha\beta\gamma}(k)\int d^3j\, d\tau\, M_{\beta\beta'\gamma'}(j)G_0(j,t,\tau)\times D_{\gamma'\alpha}(k)U(|\mathbf{k}-\mathbf{j}|t-\tau)\times D_{\beta'\gamma}(k-j) \tag{34}$$

in which the leading-order subgrid autocorrelation

$$\langle u_\alpha^>(\mathbf{k},t)u_\beta^>(\mathbf{k}',t')\rangle=D_{\alpha\beta}(k)U(|\mathbf{k}|,t-t')\delta(\mathbf{k}+\mathbf{k}') . \tag{35}$$

For free-decay turbulence U is related to the energy spectrum [Eqs. (22) and (25)], while for forced turbulence U is given through the correlation function [Eqs. (16) and (17)].

One now proceeds to eliminate the next subgrid shell. For the quadratic nonlinearity in Eq. (33) one proceeds as in the first iteration. However, since the triplet nonlinearity in Eq. (33) is already $O(\lambda^2)$ there will be no difference in the way this term is handled—either by the traditional multiple scale elimination procedure [10,11] or by the present multiple scale smoothing procedure. One readily finds form invariance after the second iteration, so that at the $(n+1)$th iteration the resolvable scale momentum equation takes the form

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha(\mathbf{k},t)+\int_{-\infty}^{t} d\tau \sum_{m=0}^{n} \eta_m(k,t,\tau)u_\alpha(\mathbf{k},\tau)$$
$$=f_\alpha(\mathbf{k},t)+\lambda M_{\alpha\beta\gamma}(k)\int d^3j\left[u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)+2\lambda\int d\tau\, d^3j' M_{\beta\beta'\gamma'}(j)\sum_{m=0}^{n} G_m(j,t,\tau)u_{\beta'}(\mathbf{j}',\tau)\right.$$
$$\left.\times u_{\gamma'}(\mathbf{j}-\mathbf{j}',\tau)u_\gamma(\mathbf{k}-\mathbf{j},t)\right] , \tag{36}$$

with the Green's function G_m satisfying

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right] G_m(k,t,\tau)+\int_{-\infty}^{t} ds\, G_m(k,s,\tau)\times\eta_{m-1}(k,s,\tau)=\delta(t-\tau) . \tag{37}$$

The eddy-damping function η_m consists of contributions from both the quadratic and triplet nonlinearities:

$$\eta_m(k,t,\tau)=\eta_m^D(k,t,\tau)+\eta_m^T(k,t,\tau) , \tag{38}$$

where

$$\eta_m^D(k,t,\tau)=-4M_{\alpha\beta\gamma}(k)\int d^3j\, d\tau\, M_{\beta\beta'\gamma'}(j)G_m(j,t,\tau)\times D_{\gamma'\alpha}(k)U(|\mathbf{k}-\mathbf{j}|,t-\tau)\times D_{\beta'\gamma}(j-j) \tag{39}$$

and the triplet interactions yield

$$\eta_m^T(k,t,\tau)=-4M_{\alpha\beta\gamma}(k)\int d^3j\,d\tau\,M_{\beta\beta'\gamma'(j)}\times\sum_{m'=0}^{m-1}G_{m'}(j,t,\tau)D_{\gamma'\alpha}(k)\times U(|\mathbf{k}-\mathbf{j}|,t-\tau)\times D_{\beta'\gamma}(k-j)\ . \quad (40)$$

Equations (36)–(40) are the required RNG momentum equations for the resolvable scales. These equations are identical to those derived earlier for forced turbulence using the standard multiple scale elimination approach [11] but are a generalization of our Markovian free-decaying turbulence theory [10] due to the presence of the nonlocal behavior of the eddy-damping function. In Appendix A we shall show that this non-Markovian forced and free-decay result can also be obtained from the multiple scale elimination approach. Thus we have now provided a formalism that is not only independent of the order in which the subgrid average is performed but which shows that the quartic resolvable scale interactions $u^<u^<u^<u^<$ will not appear to $O(\lambda^2)$ in the recursive-RNG momentum equation. In earlier formulations [10,11] these fourth-order nonlinearities were simply dropped without proper justification.

III. EFFECTS OF THE TRIPLE NONLINEARITIES IN THE RENORMALIZED NAVIER-STOKES EQUATION ON THE EDDY VISCOSITY

A. Introduction

We now turn to the important question: what are the effects of these triple-order nonlinearities on the eddy viscosity? In free-decay turbulence, we presented only a qualitative argument [10] that these triple nonlinearities will contribute an extra damping effect in the momentum equation near k_c. In the case of forced turbulence, we simplified [11] the RNG momentum equation by taking advantage of the fact that the subgrid modes evolve on a faster time scale than the resolvable scales. To leading order one then loses the nonlocal behavior of the eddy-damping function, but the resulting integro-difference eddy-viscosity recursion relation is more tractable and leads to a fixed point, $\nu^*(k)$. Thus, in either free or forced turbulence, the RNG Navier-Stokes equation can be simplified to

$$\left[\frac{\partial}{\partial t}+\nu^*(k)k^2\right]u_\alpha(\mathbf{k},t)=f_\alpha(\mathbf{k},t)+M_{\alpha\beta\gamma}(k)\int d^3j\,u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)+2M_{\alpha\beta\gamma}(k)\sum_i\int d^3j\,d^3j'h^{-4i/3}\frac{M_{\beta\beta'\gamma'}(j)}{\nu^*(h^ij)j^2}u_{\beta'}(\mathbf{j}-\mathbf{j}',t)u_{\gamma'}(\mathbf{j}',t)u_\gamma(\mathbf{k}-\mathbf{j},t) \quad (41)$$

on setting $\lambda=1$. $O<h<1$ is the scaling factor [9–11] that measures the coarseness of the subgrid partition. The direct numerical simulation of Eq. (41) is difficult and has not been attempted.

Now closure theories [20–22,26,27] are concerned with the total energy transfer that arises from the terms on the right-hand side of Eq. (7). As a result, these theories cannot identify the individual interactions that contribute to the energy transfer. However, recent advances in large-scale computations have allowed some identification.

We perform a somewhat indirect test on the role of the triple nonlinear terms in Sec. IV by analyzing high-resolution decaying [36] and forced [25,26] databases on a $128\times128\times128$ mesh. We shall show, in Sec. IV, that the $u^>u^>$ term on the right-hand side in Eq. (7) represents energy transfer consistent with subgrid eddy damping. Near the cutoff region k_c, the energy transfer arises from the energy-conserving $u^<u^<$ term in Eq. (7). The cross $u^>u^<$ term in Eq. (7) will be shown to play a key role in the removal of energy locally from k_c as well as contributing to the strong cusp behavior of the eddy viscosity around k_c. One can then see the importance of the triple nonlinearities in the recursive-RNG approach—these cubic interactions arising from the $u^>u^<$ term in Eq. (7) and which are absent from ϵ RNG.

B. Eddy-viscosity cusp behavior from energy transfer in LES [21]

In this section we shall briefly review the LES calculation of Leslie and Quarini [21] and their simulations which show that in the energy-transfer equation the eddy viscosity exhibits a strong cusp behavior as $k\to k_c$. This will be contrasted, in the following section, to what the recursive-RNG theory—with its triple nonlinearity—predicts for the eddy-viscosity behavior in the energy-transfer equation.

In the LES approach to turbulence, a filter function [14,18,21,37] F is introduced to smooth the turbulent fields (which contains information on all scales up to the dissipation wave number k_0)

$$u_\alpha(\mathbf{x},t)\equiv\overline{u_\alpha}(\mathbf{x},t)+u'_\alpha(\mathbf{x},t)\ , \quad (42)$$

where the mean (filtered) part of u_α is defined by

$$\overline{u_\alpha}(\mathbf{x},t)=\int d^3x' F(\mathbf{x}-\mathbf{x}',a)u_\alpha(\mathbf{x}',t) \quad (43)$$

and u'_α is the fluctuation. The filter function F, with characteristic length a, satisfies

$$\int d^3x' F(\mathbf{x}-\mathbf{x}',a)=1 \ , \quad \text{with} \lim_{a\to 0}\int d^3x' F(\mathbf{x}-\mathbf{x}',a)u_\alpha(\mathbf{x}')=u_\alpha(\mathbf{x}) \ . \quad (44)$$

A standard filter function used in LES is the spherically symmetric sharp cutoff special filter, which in wave-number representation is

$$F(\mathbf{k},a)=\begin{cases}1 & \text{for } k\le k_c \\ 0 & \text{for } k>k_c \ .\end{cases} \quad (45)$$

On filtering the Navier-Stokes equation, one encounters a closure problem in having to determine a closed expression for the tensor $\overline{u_\alpha u_\beta}$. Using Eq. (42),

$$\overline{u_\alpha u_\beta}=\overline{u_\alpha}\,\overline{u_\beta}+L_{\alpha\beta}+Q_{\alpha\beta} \ , \quad (46)$$

where $L_{\alpha\beta}$ is the Leonard stress

$$L_{\alpha\beta}\equiv\overline{\overline{u_\alpha}\,\overline{u_\beta}}-\overline{u_\alpha}\,\overline{u_\beta} \quad (47)$$

and $Q_{\alpha\beta}$ is the true subgrid stress

$$Q_{\alpha\beta}=\overline{u'_\alpha\overline{u_\beta}}+\overline{\overline{u_\alpha}u'_\beta}+\overline{u'_\alpha u'_\beta} \ . \quad (48)$$

The major issue in the filtering approach to LES is how to model the true subgrid stress $Q_{\alpha\beta}$ in terms of the filtered velocity and its derivatives. In the classical closure schemes, one typically works with the second-moment equations rather than the filtered Navier-Stokes equation. In particular, with random forcing, Leslie and Quarini [21] have derived the filtered energy transport equation

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]\overline{E}(k,t)=T^D(k,t)+T^S(k,t)+R \ , \quad (49)$$

where the filtered energy $\overline{E}(k,t)$ is defined by

$$\overline{E}(k,t)=\tfrac{1}{2}k^2\int d\Omega\langle\overline{u_\alpha}(\mathbf{k},t)\overline{u_\alpha}(-\mathbf{k},t)\rangle \ . \quad (50)$$

The ensemble averaging $\langle\ \rangle$ is introduced because of the random forcing, and $d\Omega$ is the angular k integration. R is the energy transfer due to the random forcing and T^D is the standard inertial transfer energy. T^S is the energy transfer due to the subgrid stresses and is the term of immediate interest. Using classical closures, Leslie and Quarini [21] show that the subgrid stress energy transfer

$$T^S(k,t)=2k^2\nu_d(k)\overline{E}(k)-\Phi(k,t) \ , \quad (51)$$

where $\nu_d(k)$ is the drain part of the eddy viscosity

$$\nu_d(k)=\sum_{p,q}A(k,p,q)[1-F(p)F(q)]E(q) \quad (52)$$

and

$$\Phi(k)=\sum_{p,q}B(k,p,q)F^2(k)[1-F(p)F(q)]E(q)E(p) \ . \quad (53)$$

F is the filter function and the total scalar energy $E(k)$ is related to the filtered energy

$$\overline{E}(k)=F^2(k)E(k) \ . \quad (54)$$

The explicit [21] functional forms of A and B are not needed here.

If one invokes the standard eddy-viscosity assumption then the subgrid stress energy transfer can be defined

$$T^S(k)=2k^2\nu_c(k)E(k) \ , \quad (55)$$

where the classical closure eddy viscosity $\nu_c(k)$ is defined by

$$\nu_c(k)=\nu_d(k)-\nu_b(k) \ . \quad (56)$$

The backscatter eddy viscosity $\nu_b(k)$ is defined by

$$\nu_b(k)=\frac{\Phi(k)}{2k^2E(k)} \ . \quad (57)$$

From the numerical work of Leslie and Quarini [21] for the sharp-cutoff filter, one finds that

$$\nu_c(k)\approx\nu_d(k) \quad \text{for } k\ll k_c \quad (58)$$

since for small k, the backscatter eddy viscosity $\nu_b(k)\approx 0$. As $k\to k_c$, $\nu_d(k)$ increases rapidly so that there will be an approximate cancellation between $\nu_d(k)$ and $\nu_b(k)$. The net effect on the classical closure eddy viscosity is that $\nu_c(k)$, Eq. (56), exhibits a strong cusp behavior as $k\to k_c$. This cusplike behavior has also been found in the test field model of Kraichnan [20], in the eddy-damped quasinormal Markovian (EDQNM) theory of Chollet and Lesieur [22], in the direct numerical simulations of Domaradzski *et al.* [23], as well as in the recent LES work of Lesieur and Rogallo [24] on passive scalar turbulence.

If the Kolmogorov inertial energy range is assumed to extend to $k\to 0$, then $\nu_c(k)$ exhibits an integrable singularity [20,21] as $k\to k_c$. If one assumes a production-type spectrum [21] (which is zero at $k=0$, peaks at wave number $k_{\max}$, and joins onto the Kolmogorov spectrum for $k>k_{\max}$) then this singularity is removed. In this case, $\nu_c(k)$ still exhibits a marked cusplike behavior as $k\to k_c$, even when the production energy spectrum peaks at $k_{\max}/k_c=0.1$.

C. Eddy-viscosity cusp behavior from energy transfer in recursive RNG

Consider the RNG momentum equation (41). This equation can be simplified by passing to the differential partition limit [9] $h\to 1$, in which

$$\frac{j}{k_c}h^i\to 1, \quad \nu^*(h^i j)\to\nu^*(k_c), \quad h^{-4i/3}\to\left[\frac{j}{k_c}\right]^{4/3} . \quad (59)$$

In this limit, Eq. (41) reduces to

$$\left[\frac{\partial}{\partial t}+\nu^*(k)k^2\right]u_\alpha(\mathbf{k},t)=f_\alpha(\mathbf{k},t)+M_{\alpha\beta\gamma}(k)\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t) +2M_{\alpha\beta\gamma}(k)\int d^3j\, d^3j'\left[\frac{j}{k_c}\right]^{4/3}\frac{M_{\beta\beta'\gamma'}(j)}{\nu^*(k_c)j^2}u_\beta(\mathbf{j}-\mathbf{j}',t)u_{\gamma'}(\mathbf{j}',t)u_\gamma(\mathbf{k}-\mathbf{j},t)\ , \tag{60}$$

where $\nu^*(k)$ is the appropriate RNG eddy viscosity [10,11] and is discussed in some detail in Appendix C. The various wave-number ranges for the last term of Eq. (60) are

$$|\mathbf{k}|<k_c,\quad |\mathbf{j}'|<k_c,\quad |\mathbf{j}-\mathbf{j}'|<k_c\ ,\qquad k_c<|\mathbf{j}|<k+k_c\ . \tag{61}$$

1. *Transport equation*

The second-moment equations can be readily derived from Eq. (60) by determining the time evolution of

$$U_{\alpha\beta}(\mathbf{k},t)=\langle u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle\ , \tag{62}$$

i.e.,

$$\frac{\partial U_{\alpha\beta}(\mathbf{k},t)}{\partial t}=-2\nu^*(k)k^2U_{\alpha\beta}(\mathbf{k},t)+2\langle f_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle +T^D_{\alpha\beta}(\mathbf{k},t)+T^T_{\alpha\beta}(\mathbf{k},t)\ , \tag{63}$$

where T^D is the usual energy-transfer term due to the quadratic nonlinearity in Eq. (60),

$$T^D_{\alpha\beta}=2M_{\alpha\beta'\gamma}(k)\int d^3j\langle u_{\beta'}(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)u_\beta(-\mathbf{k},t)\rangle\ , \tag{64}$$

while T^T is the energy transfer arising from the triple nonlinearity induced by RNG,

$$T^T_{\alpha\beta}=4M_{\alpha\beta'\gamma}(k)\int d^3j\, d^3j'\frac{M_{\beta'\gamma'\kappa}(j)}{\nu^*(k_c)j^2}\left[\frac{j}{k_c}\right]^{4/3}\langle u_{\gamma'}(\mathbf{j}-\mathbf{j}',t)u_\kappa(\mathbf{j}',t)u_\gamma(\mathbf{k}-\mathbf{j},t)u_\beta(-\mathbf{k},t)\rangle\ . \tag{65}$$

Thus T^T is a term arising in Eq. (63) which is not present in the LES theory of Leslie and Quarini [21] or the ϵ-RNG theory of Dannevik, Yakhot, and Orzsag [38].

To leading order, T^T can be simplified by applying the quasinormal approximation to the fourth-order moment in Eq. (65),

$$\langle u_{\beta'}(\mathbf{j}-\mathbf{j}',t)u_\kappa(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)u_\beta(-\mathbf{k},t)\rangle \approx U_{\beta'\kappa}(\mathbf{j}-\mathbf{j}')U_{\gamma\beta}(\mathbf{k}-\mathbf{j})\delta(\mathbf{j})+U_{\beta'\gamma}(\mathbf{j}-\mathbf{j}')U_{\kappa\beta}(\mathbf{j}')\delta(\mathbf{k}-\mathbf{j}')+U_{\beta'\beta}(\mathbf{j}-\mathbf{j}')U_{\kappa\gamma}(\mathbf{j}')\delta(\mathbf{k}+\mathbf{j}'-\mathbf{j})\ . \tag{66}$$

This is a standard approximation and has been employed successfully in a variety of physical problems [39–41], and in particular in the context of ϵ RNG [38]. It is justified here because it is made in expressions that are dependent on the eddy viscosity [and not the molecular viscosity, see Eq. (67)]. Another justification is that to leading order $u^{>0}$ has Gaussian properties [see Eqs. (11)–(13) and the expected asymptotic unimportance of any non-Gaussian components in the forcing statistics [38]]. The first term in Eq. (66) cannot contribute to T^T due to the $\delta(\mathbf{j})$ factor and the fact that the $\int d^3j$ integration has $|\mathbf{j}|$ restricted to the shell $k_c<j<k+k_c$. Hence, on substituting Eq. (66) into (65), one obtains

$$T^T_{\alpha\beta}=8M_{\alpha\beta'\gamma}(k)\int d^3j\frac{M_{\beta'\gamma'\kappa}(j)}{\nu^*(k_c)j^2}\left[\frac{j}{k_c}\right]^{4/3}\times U_{\gamma'\gamma}(\mathbf{j}-\mathbf{k})U_{\kappa\beta}(\mathbf{k})\ . \tag{67}$$

The energy spectrum $E(\mathbf{k},t)$ is related to the correlation function $U_{\alpha\beta}$,

$$E(\mathbf{k},t)=2\pi k^2D_{\alpha\beta}(\mathbf{k})U_{\alpha\beta}(\mathbf{k},t)\ , \tag{68}$$

where $D_{\alpha\beta}$ is the projection operator, Eq. (4), so that its time evolution is

$$\left[\frac{\partial}{\partial t}+2\nu^*(k)k^2\right]\frac{E(\mathbf{k},t)}{4\pi k^2} =T^D_{\alpha\alpha}+T^T_{\alpha\alpha}+\langle f_\alpha(\mathbf{k},t)u_\alpha(-\mathbf{k},t)\rangle\ . \tag{69}$$

For forced turbulence, the covariance [see Eq. (13)]

$$U_{\alpha\beta}(\mathbf{k})=\frac{D_0D_{\alpha\beta}(\mathbf{k})}{2\nu^*(k)k^2}k^{-y} \tag{70}$$

will recover the Kolmogorov energy spectrum $k^{-5/3}$ for the exponent $y=3$. Following Leslie and Quarini [21], we can define a drain-eddy viscosity ν_T associated with this triple nonlinearity energy transfer T^T,

$$T^T_{\alpha\alpha} \equiv -2\nu_T(k)k^2 \frac{E(k)}{4\pi k^2} , \tag{71}$$

so that

$$\nu_T(k) = \frac{1}{2\nu^*(k_c)k_c^{4/3}} \frac{1}{k^2} \times \int_{k_c}^{k+k_c} dj \int_{-1}^{1} d\mu \frac{L_{kj} D_0 |\mathbf{k}-\mathbf{j}|^{-y-2} j^{4/3}}{\nu(|\mathbf{k}-\mathbf{j}|)} , \tag{72}$$

where $\mathbf{k}\cdot\mathbf{j} = kj\mu$, and L_{kj} is given by

$$L_{kj} = -\frac{kj(1-\mu^2)[\mu(k^2+j^2) - kj(1+2\mu^2)]}{k^2+j^2-2kj\mu} . \tag{73}$$

The eddy viscosity $\nu^*(k_c)$, appearing in the denominator of Eq. (72), is the renormalized viscosity of the recursive-RNG Navier-Stokes equation [10,11] so that ν_T reduces to

$$\nu_T(k) = \frac{1}{2\nu^*(k_c)k_c^{4/3}} \frac{1}{k^2} \times \int_{k_c}^{k+k_c} dj \int_{-1}^{1} d\mu \frac{L_{kj} j^{4/3}}{0.1904 \varepsilon^{1/3} K_0^2 |\mathbf{k}-\mathbf{j}|^{11/3}} . \tag{74}$$

From recursive-RNG calculations [11] it is shown in Appendix C that $\nu^*(k_c) = 0.329$, while the renormalized eddy viscosity $\nu(|\mathbf{k}-\mathbf{j}|)$ in the integrand of Eq. (2) takes the form [11,42]

$$\nu(|\mathbf{k}-\mathbf{j}|) = 0.1904 K_0^2 \varepsilon^{1/3} k^{-4/3} . \tag{75}$$

K_0 is the Kolmogorov constant and ε is the total rate of dissipation of energy in the subgrid scales.

From Eq. (74), one can readily show that the drain-eddy viscosity $\nu_T(k) < 0$ for $k/k_c \ll 1$. Indeed, since $|\mathbf{k}-\mathbf{j}|$ lies in the resolvable scale, $|\mathbf{k}-\mathbf{j}| < k_c$. Thus, for $k \ll k_c$, the effect of the restrictions on the j integration

$$k\mu < j < k_c + k\mu \quad \text{and} \quad k_c < j < k_c + k$$

results in the integration domain

$$0 \le \mu \le 1, \quad k_c < j < k_c + k\mu .$$

Using Eq. (73) and some straightforward expansions, we find for $k \ll k_c$

$$\nu_T(k) = \frac{1}{2\nu^*(k_c)k_c^{4/3}} \frac{1}{k^2} \times \int_0^1 d\mu \int_{k_c}^{k_c+k\mu} dj \frac{L_{kj} j^{4/3}}{0.1904 \varepsilon^{1/3} K_0^2 |\mathbf{k}-\mathbf{j}|^{11/3}}$$
$$\approx -\frac{1}{2.856 \varepsilon^{1/3} K_0^2} \frac{1}{\nu^*(k_c) k_c^{8/3}} + O\left[\frac{k}{k_c}\right] . \tag{76}$$

The fact that the drain-eddy viscosity associated with the RNG-induced triple nonlinearities is negative for $k \ll k_c$ implies that there is a backscatter of energy from the small scale to the large spatial scales. This backscatter effect has also been found in the recent numerical work of Chasnov [25,26] as well as in the numerical work of Domaradzski *et al.* [23].

2. *Determination of the drain-eddy viscosity associated with the triple nonlinearity*

Consider a production-type energy spectrum [21] of the form

$$E(k) = A_S \left[\frac{k}{K_p}\right] K_0 \varepsilon^{2/3} k^{-5/3} , \tag{77}$$

where the function $A_s(x)$ is

$$A_s(x) = \frac{x^{s+5/3}}{1+x^{s+5/3}} \quad \text{for exponent } s , \tag{78}$$

K_p is a constant which is directly correlated to the location of the maximum in $E(k)$. From Eq. (78) we find that

$$E(k) \to k^s \quad \text{as } k \to 0 \tag{79}$$

so that from direct numerical simulation results [24–26] an appropriate choice for the exponent s is

$$s = 4 . \tag{80}$$

On the other hand, $A_s(k/K_p) \to 1$ for $k \gg K_p$ so that the production-type spectrum, Eq. (77), joins smoothly onto the standard Kolmogorov spectrum. In Fig. 1, the production energy spectrum, Eqs. (77) and (78), is plotted for $s = 4$ and for various values of K_p. Note that as K_p decreases, the peak in the energy spectrum moves to smaller k.

For the case of forced turbulence we plot, in Fig. 2, the drain-eddy viscosity $\nu_T(k)$ associated with the triple nonlinearity transfer function T^T for the production-type spectrum $E(k)$ for various values of the parameter

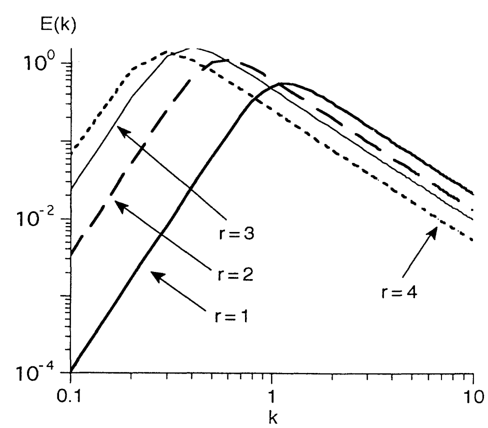

FIG. 1. The production energy spectrum $E(k)$, Eqs. (77)–(80), for various values of $r = k_c/K_p$. These $E(k)$ curves are normalized so that $\int dk E(k) = $const, independent of r.

$r=k_c/K_p$. Note the appearance of a strong cusp in the drain-eddy viscosity $\nu_T(k)$ as $k\to k_c$ for $r>1$. Also plotted in Fig. 2 is the renormalized eddy viscosity $\nu^*(k)$ that arises in the renormalization of the Navier-Stokes equation and has been determined in earlier calculations [11]. $\nu^*(k)$ is insensitive to the value of the parameter r. As expected from the small-k asymptotic result of Eq. (76), the drain-eddy viscosity $\nu_T(k)<0$ and over a considerable wave-number range: $k/k_c<0.45$. As mentioned earlier, this negative drain-eddy viscosity represents a back transfer of energy from small scales to large spatial scales and has also been seen in recent numerical simulations [23,25,26].

In Fig. 3, for free-decaying turbulence [10], we plot the analogous drain-eddy viscosities $\nu_T(k)$ as well as the corresponding renormalized momentum eddy viscosity $\nu^*(k)$. Again, $\nu_T(k)<0$ for small k so that there is again a backscatter of energy from small scales to large spatial scales. A strong cusp feature is also present as $k\to k_c$.

The net eddy viscosity

$$\nu_{\rm net}(k)=\nu^*(k)+\nu_T(k) \tag{81}$$

is plotted in Figs. 4 and 5 for forced and free-decaying turbulence, respectively. For $r>1$, we again find a strong cusp feature in the net eddy viscosity as $k\to k_c$. This is to be compared with the net eddy viscosity found from the closure theories of Kraichnan [20], Leslie and Quarini [21], and Chollet and Lesieur [22] which are also shown in Figs. 4 and 5. Note that for $r=1$, the inertial $k^{-5/3}$ Kolmogorov spectral form occurs only in the range $k/k_c>1$ (see Fig. 1) and so the case of $r=1$ does not correspond directly with any numerical or LES simulation results. This somewhat unphysical case is included to show the effect of r on the strength of the cusp as $k\to k_c$.

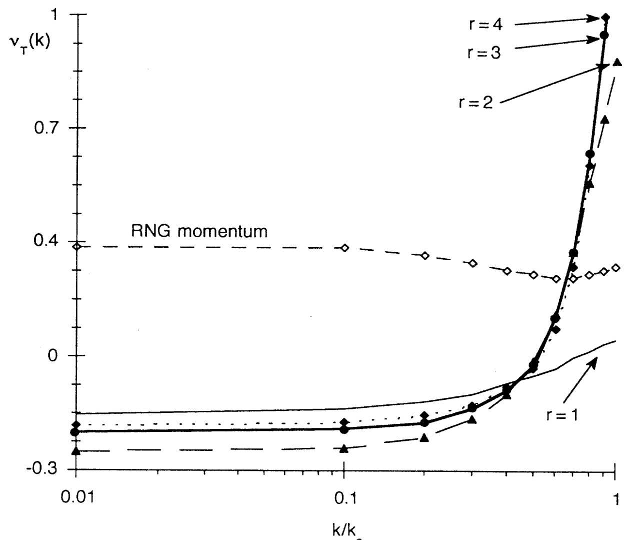

FIG. 3. The drain-eddy viscosity $\nu_T(k)$ arising from the triple nonlinearities for the case of free-decaying turbulence. $r=k_c/K_p$. Note that $\nu_T(k)<0$ for $k/k_c<0.45$ but this energy backscatter is not as pronounced as in the case of forced turbulence. Also the cusp behavior in $\nu_T(k)$ as $k\to k_c$ is somewhat milder. For comparison, the (free-decay) renormalized $\nu^*(k)$ arising in the RNG momentum equation is also plotted.

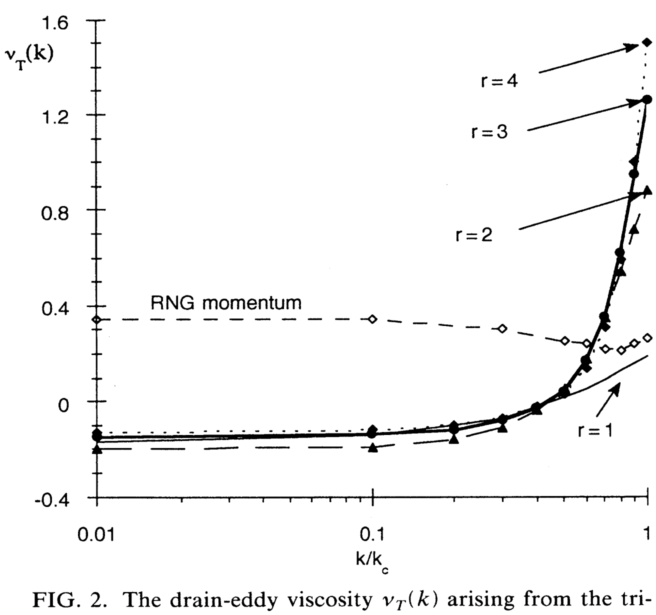

FIG. 2. The drain-eddy viscosity $\nu_T(k)$ arising from the triple nonlinearities for the case of forced turbulence. $r=k_c/K_p$. Note that $\nu_T(k)<0$ for $k/k_c<0.45$. This is indicative of backscatter of energy from the subgrid scales to the large scales and this effect has been seen in direct numerical simulations [23,25,26]. Note the sharp cusp behavior in $\nu_T(k)$ as $k\to k_c$. For comparison, the renormalized $\nu^*(k)$ arising in the RNG momentum equation is also plotted.

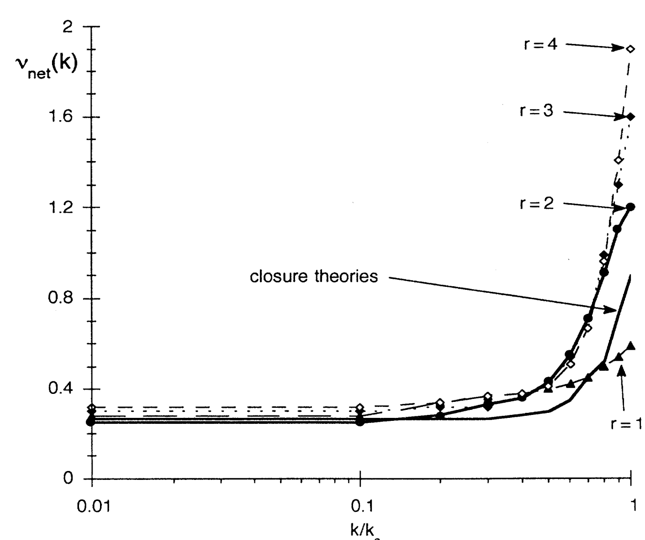

FIG. 4. A comparison of the net eddy viscosity $\nu_{\rm net}(k)=\nu_T(k)+\nu^*(k)$ arising in the RNG energy transport equation with that arising from closure theories for forced turbulence. $r=k_c/K_p$.

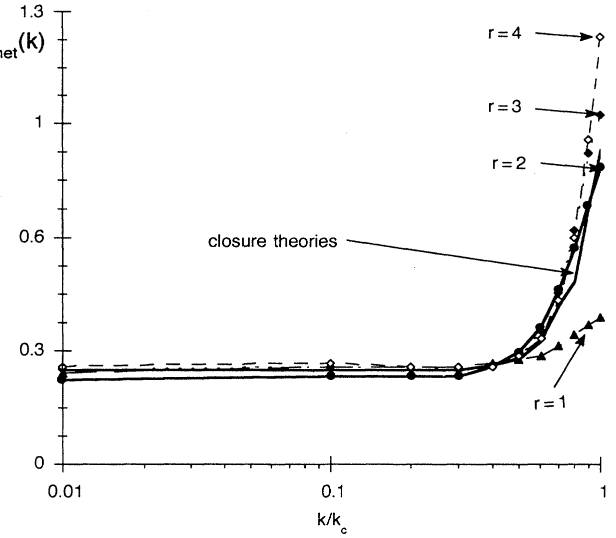

FIG. 5. A comparison of the net eddy viscosity $\nu_{\rm net}(k)=\nu_T(k)+\nu^*(k)$ arising in the RNG energy transport equation with that arising from closure theories for free-decaying turbulence. $r=k_c/K_p$.

IV. SUBGRID EDDY VISCOSITY FROM NUMERICAL SIMULATION DATABASES

In numerical simulations, energy transfer and eddy viscosity are analyzed by introducing an artificial cut $k_c<k_m$, where k_m is the highest resolvable wave number in the simulation. Thus the subgrid scales have $k_c<k<k_m$ while the resolvable modes have $k<k_c$. With this wave-number separation, it is possible to evaluate explicitly the effect of the subgrid scales on the resolvable modes. It is convenient to rewrite the energy-transfer equation for the resolvable scales, readily derivative from Eq. (7), in the form

$$\left[\frac{\partial}{\partial t}+2\nu_0 k^2\right]E(k,t)=T^{<<}(k)+T^{><}(k)+T^{>>}(k)\ , \quad (82)$$

where $k<k_c$, and

$$T^{<<}(k)=\mathrm{Im}\left[M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\alpha^{<*}(\mathbf{k})u_\beta^{<}(\mathbf{j},t)\times u_\gamma^{<}(\mathbf{k}-\mathbf{j},t)\right] \quad (83)$$

gives the rate of energy transfer to mode $\mathbf{k}$ from interactions between the resolvable scales $u_\beta^<$ and $u_\gamma^<$ (we have reverted back to the notation of $u^<$ for the resolvable modes and $u^>$ for the subgrid modes),

$$T^{><}(k)=\mathrm{Im}\left[M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\alpha^{<*}(\mathbf{k})2u_\beta^{>}(\mathbf{j},t)\times u_\gamma^{<}(\mathbf{k}-\mathbf{j},t)\right] \quad (84)$$

is the energy-transfer rate to mode $\mathbf{k}$ from the interactions between resolvable-subgrid modes $u^>u^<$, and

$$T^{>>}(k)=\mathrm{Im}\left[M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\alpha^{<*}(\mathbf{k})u_\beta^{>}(\mathbf{j},t)u_\gamma^{>}(\mathbf{k}-\mathbf{j},t)\right] \quad (85)$$

is that transfer rate to mode $\mathbf{k}$ from the interactions between subgrid-subgrid modes $u^>u^>$. Since the effects of subgrid modes are conventionally modeled by eddy viscosities, we thus define

$$\nu^{><}(k|k_c)\equiv-\frac{T^{><}(k)}{2k^2E(k)},\quad \nu^{>>}(k|k_c)\equiv-\frac{T^{>>}(k)}{2k^2E(k)}\ , \quad (86)$$

with the total spectral eddy viscosity being given by

$$\nu_{\rm tot}(k|k_c)=\nu^{>>}(k|k_c)+\nu^{><}(k|k_c)\ . \quad (87)$$

The eddy viscosities in Eq. (86) are determined directly from numerical simulations by calculating the modal energy transfer from velocity field databases generated on 128^3 meshes for both forced and free-decaying Navier-Stokes turbulence. The wave-number cutoff k_c is chosen at $k_c=32$. The forced flow database was generated by Chasnov [25] in a LES of the Kolmogorov inertial range using a subgrid model derived from a stochastic equation that is consistent with EDQNM theory. The free-decay DNS database was generated by Lee [36] at a microscale Reynolds number of about 50. Our analysis is performed at one fixed time instant—so that, in essence, we are examining one member from an ensemble of realizations.

In the stimulated inertial range, the eddy viscosities are plotted in Figs. 6 and 7 for forced and free-decaying turbulence, respectively. For forced turbulence, the eddy viscosity is positive for small k (Fig. 6), but for free-decaying turbulence the eddy viscosity can become negative [23] for a substantial range of small k if the wave-number cutoff k_c lies in the far dissipation range [20]. If k_c is chosen closer to the energy containing range then the wave-number region of negative eddy viscosity is restricted to the smallest k's (Fig. 7). The free-decay eddy viscosity exhibits features similar to that found in the forced-turbulence case as $k\to k_c$.

Consider first $\nu^{>>}(k)$, which arises from the subgrid-subgrid interactions. From Figs. 6 and 7 we see that $\nu^{>>}(k)\to$const for small k. This shows that the modeling of these interactions results in an eddy viscosity in direct analogy with the concept of molecular viscosity. Note moreover that $\nu^{>>}(k)$ decreases as $k\to k_c$ so that $\nu^{>>}(k)$ cannot give rise to the sharp cusp around k_c, as seen in DNS and LES. It should also be remembered that if a spectral gap approximation is made, then the only contribution to $\nu_{\rm tot}(k)$ arises from $\nu^{>>}(k)$.

On the other hand, we find from the analysis of both forced and free-decaying turbulence databases at a given time instant, that the contribution of the resolvable-subgrid interactions will give rise to $\nu^{><}(k)$ which exhibits a sharp cusp as $k\to k_c$. Our recursive-RNG eddy viscosity, Eq. (81) and Figs. 4 and 5, is in excellent agreement with the results of these LES and DNS databases,

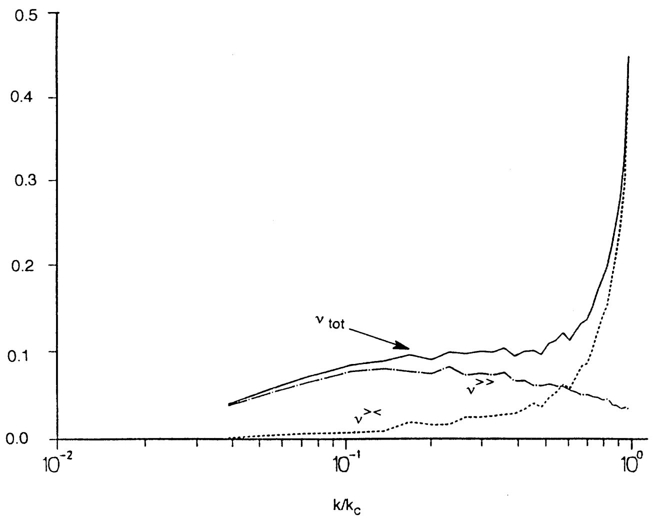

FIG. 6. Forced-eddy viscosity contributions as determined directly from LES databases for the fluid velocity at one time instant. $\nu_{\rm tot}(k)=\nu^{\gg}(k)+\nu^{><}(k)$, where $\nu^{\gg}(k)$ arises from measured LES nonlocal subgrid energy transfer and $\nu^{><}(k)$ arises from measured LES local subgrid energy transfer. Note the strong cusp arising from the local subgrid energy transfer as $k \to k_c$. It should be noted that when one wishes to compare LES results with EDQNM results one must scale the LES results by a factor of about 1.7 to account for subgrid-scale transfers k_c as a result of interactions across k_m (the interested reader should consult Refs. [24–26] for more details).

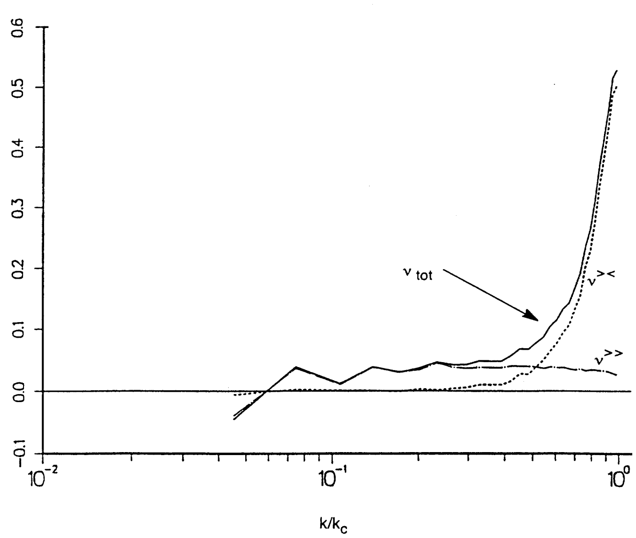

FIG. 7. Free-eddy viscosity contributions as determined directly from DNS databases for the fluid velocity at one time instant. $\nu_{\rm tot}(k)=\nu^{\gg}(k)+\nu^{><}(k)$, where $\nu^{\gg}(k)$ arises from measured DNS nonlocal subgrid energy transfer and $\nu^{><}(k)$ arises from measured DNS local subgrid energy transfer. Note the strong cusp arising from the local subgrid energy transfer as $k \to k_c$. These simulations are at low Reynolds number so that there is not an extended inertial range (i.e., we are in a situation similar to the case $r=1$ in Fig. 1).

Figs. 6 and 7. It is precisely this interaction between resolvable-subgrid scales that gives rise to the cubic nonlinearities in the recursive-RNG formulation. Without the inclusion of these higher-order nonlinearities there would be no cusp in the renormalized eddy viscosity.

In Figs. 8 and 9, we examine how the transfer spectra are affected by the various modal interactions as determined from the LES and DNS Navier-Stokes databases. The resolvable-resolvable interactions yield an energy-transfer term $T^{\ll}$ that conserves total energy within the resolvable range, but within this resolvable range it moves energy to higher wave numbers—essentially to the next octave. The subgrid-subgrid transfer $T^{\gg}$ removes energy throughout the resolvable scales in a manner consistent with the concept of an eddy viscosity. But the resolvable-subgrid transfer $T^{><}$ primarily removes energy from the last resolvable octave that had been transferred there by $T^{\ll}$ so that there is a local flow of energy through k_c. From Figs. 8 and 9, one can see that the transfer $T^{><}$ is most important near k_c, and it accounts for most of the energy flow from the resolvable scales, consistent with the corresponding cusp behavior of $\nu^{><}$ as $k \to k_c$.

Thus our analysis of subgrid-scale energy transfer and eddy viscosities from numerical simulation databases of the flow field for both forced turbulence at high Reynolds numbers and for free-decaying turbulence at low Reynolds numbers indicates that it is not possible to obtain local energy flow through the wave-number cutoff k_c if one only considers subgrid-subgrid interactions. This local energy flow has been shown to be directly related to the cross term $u^{>}u^{<}$ in Eq. (7), a term discarded in the ϵ-RNG approach of Yakhot and Orszag [7].

A direct numerical simulation of the recursive-RNG momentum equation, Eq. (41), is quite prohibitive—both

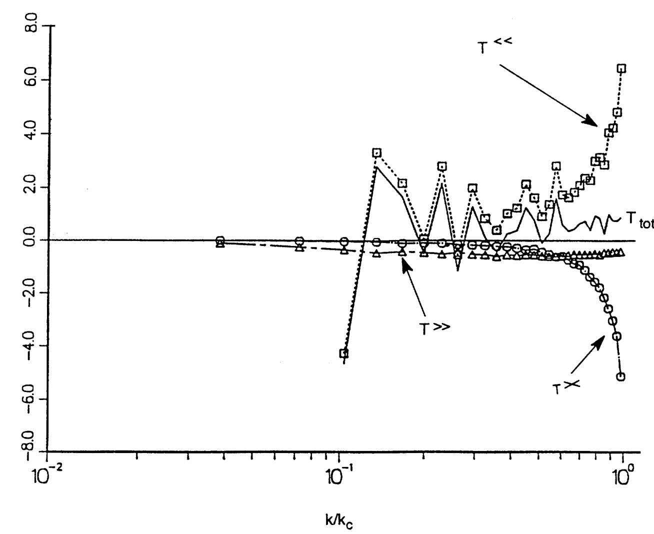

FIG. 8. Measured LES energy transfers corresponding to the forced eddy viscosities in Fig. 6, together with the resolvable scale transfer $T^{\ll}(k)$. $T^{\gg}(k)$ is the nonlocal subgrid transfer while $T^{><}(k)$ is the local subgrid transfer.

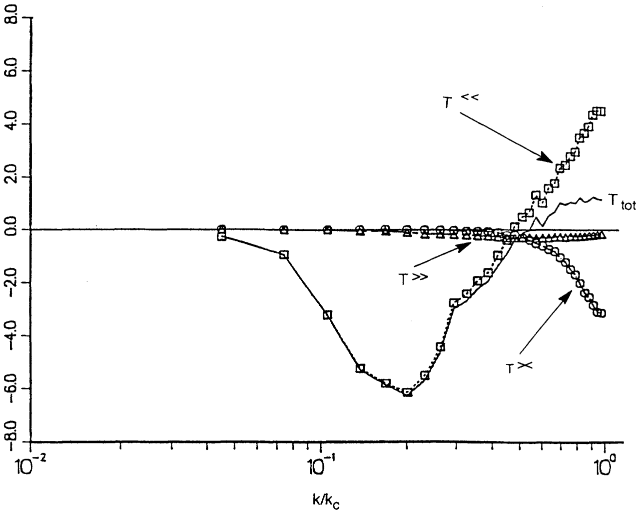

FIG. 9. Measured DNS energy transfers corresponding to the free-decaying eddy viscosities in Fig. 7, together with the resolvable scale transfer $T^{<<}(k)$. $T^{>>}(k)$ is the nonlocal subgrid transfer while $T^{><}(k)$ is the local subgrid transfer.

in computational effort and storage, due to the cubic nonlinearities. Recently, Hossain [43] has attempted to directly model the effect of these RNG-induced cubic nonlinearities in a low-resolution (on 32^3 and on 64^3 grids) forced-turbulence simulation. Even the modeling of these cubic nonlinearities requires substantial computational effort. However, a full Kolmogorov $k^{-5/3}$ energy spectrum was recovered [43] in these simulations, even up to k_c. If the effects of the cubic interactions were ignored, then the energy spectrum exhibited a flattening or even a rise near k_c. This phenomenon is also clearly seen in the high-resolution free-decay simulations of Chasnov [25,26]. This again illustrates the importance of retaining these cubic nonlinearities and that essential physics is lost by discarding these interactions.

V. CONCLUSION

In this paper we have reformulated the recursive-RNG theory [10,11] to make it invariant to the order in which the subgrid averages are performed—a property that is to be expected on physical grounds. We find that one still obtains the same nonlocal eddy damping functions and triple nonlinearities in the renormalized momentum equation as before—but we present here a perturbation formalism which shows that (i) the triple nonlinearity occurs at the same order as the eddy viscosity, while (ii) fourth-order nonlinearities are higher order than the triple nonlinearities—and since we have already neglected all such higher-order terms, these fourth-order nonlinearities must also be neglected for consistency. Since the perturbation parameter is eventually set to unity, this perturbation scheme is purely formal—but it is the basic approximation made in all RNG theories, both recursion RNG and ϵ RNG.

The triple nonlinearities in the momentum equation [10,11] exhibited a relatively weak cusp behavior in the eddy viscosity as $k \to k_c$, the wave number separating the subgrid from the resolvable scales.

Now in LES [20–22] and DNS [23,24] the role of the eddy viscosity is usually considered in the context of the energy transfer with the eddy viscosity exhibiting a strong cusplike behavior as $k \to k_c$. In this paper, we consider the effects of the triple nonlinearities on the energy transfer and show that the corresponding drain-eddy viscosity exhibits a strong cusp as $k \to k_c$, in good agreement with the LES [20–22] and DNS [23,24] calculations and by our own examination of both forced- and free-decaying turbulence simulation databases (Sec. IV). Moreover, it is shown analytically that this drain-eddy viscosity is negative for small k/k_c, in agreement with recent numerical simulations [23,25,26]. This negative drain-eddy viscosity represents a backscatter of energy from small to large spatial scales.

On the other hand, the ϵ-RNG theories [5–7] ignore the effects of these higher-order nonlinearities. While this can be justified for $\epsilon \ll 1$, the Kolmogorov energy spectrum is only recovered in the limit $\epsilon \to 4$, and in this limit there is no justification for neglecting these higher-order nonlinearities. Moreover, the eddy viscosities derived in these theories hold only in the limit $k \to 0$ and for the case of forced turbulence only. The recursive-RNG approach, on the other hand, yields a wave-number-dependent eddy viscosity which is valid for both free-decaying and forced turbulence. There are some subtleties in performing the two limit processes $h \to 1$ and $k \to 0$ in the recursive RNG, but these are of a purely technical nature and are planned to be reported elsewhere.

ACKNOWLEDGMENTS

The authors are grateful to Dr. R. S. Rogallo for his assistance with the computations reported in Sec. IV and for his constructive comments. One of us (Y.Z.) also expresses his appreciation to Professor W. C. Reynolds and Professor P. Moin for their encouragement and support. The work of G.V. was partially supported by the U.S. Department of Energy.

APPENDIX A: PERTURBATION EXPANSION ON THE MULTIPLE SCALE ELIMINATION PROCEDURE [10,11]

For completeness, it is worthwhile to consider what happens in the old recursive-RNG formulation [11] when the subgrid velocity perturbation expansion, Eq. (10), is performed. We shall outline in this appendix that (i) the resultant momentum equation is independent of the order in which the subgrid averaging is performed; (ii) the fourth-order nonlinearities are at least $O(\lambda^3)$, and are ignorable, and (iii) the consistency constraint, Eq. (26), is automatically satisfied.

If one first substitutes Eq. (11), the leading-order subgrid solution $u^{>0}$, into the resolvable scale momentum equation (7) and then performs the subgrid average, then there is no nontrivial information. The role of $u^{>0}$ is just to specify the subgrid autocorrelation.

The nontrivial contribution to the resolvable scales comes from $u^{>1}$. From Eq. (7)

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^<(\mathbf{k},t)=f_\alpha^<(\mathbf{k},t)+\lambda M_{\alpha\beta\gamma}(k)\int d^3j[u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+2u_\beta^{>1}(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+2u_\beta^{>1}(\mathbf{j},t)u_\gamma^{>0}(\mathbf{k}-\mathbf{j},t)]+O(\lambda^3)\ . \tag{A1}$$

One now substitutes Eq. (18) for $u^{>1}$ into Eq. (A1) and then performs the subgrid average. We note that at $O(\lambda^2)$ (i) the $u^{>1}u^<$ term in Eq. (A1) will generate the triple nonlinearity

$$2M_{\alpha\beta\gamma}(k)\int d^3j\,d^3j'd\tau\,G_0(j,t,\tau)M_{\beta\beta'\gamma'}(j)u_{\beta'}^<(\mathbf{j}',\tau)u_{\gamma'}^<(\mathbf{j}-\mathbf{j}',\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},\tau)\ , \tag{A2}$$

(ii) the $u^{>1}u^{>0}$ term in Eq. (A1) will generate the nonlocal eddy-damping function $\eta_0(k,t,\tau)$,

$$4M_{\alpha\beta\gamma}(k)\int d^3j\,d^3j'd\tau\,G_0(j,t,\tau)M_{\beta\beta'\gamma}(j)\langle u_{\beta'}^{>0}(\mathbf{j}',\tau)u_\gamma^{>0}(\mathbf{k}-\mathbf{j},\tau)\rangle u_{\gamma'}^<(\mathbf{j}-\mathbf{j}'\tau)\equiv-\int_{-\infty}^{t}d\tau\,\eta_0(k,t,\tau)u_\alpha^<(\mathbf{k},\tau)\ . \tag{A3}$$

Note that η_0 depends only on the leading-order subgrid autocorrelation and that Eq. (17) has been used to reduce Eq. (A3). (iii) The $u^{>1}u^{>0}$ term in Eq. (A1) will also generate a term of the form $\langle u^{>0}u^{>0}u^{>0}\rangle$, which is dropped as a result of the closure approximation.

After eliminating the first subgrid shell, the resolvable scale momentum equation is just that given by Eqs. (33) and (34). On removing all the subgrid shells, one finds that one recovers the same final set of equations as when the subgrid averages are performed first.

It should be noted that in our earlier multiple scale elimination RNG technique [11], we kept the triple nonlinearities $u^<u^<u^<$ but dropped the fourth-order nonlinearities $u^<u^<u^<u^<$ without any justification. With the perturbation expansion Eq. (10) for the subgrid scales, one can now show that these fourth-order nonlinearities $u^<u^<u^<u^<$ are in fact $O(\lambda^3)$, while the triple nonlinearities $u^<u^<u^<$ and eddy viscosity are $O(\lambda^2)$. Hence one is justified in dropping these fourth-order nonlinearities.

For free-decay turbulence, one would proceed with the resolvable scale velocity in the form

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^<(\mathbf{k},t)=\lambda M_{\alpha\beta\gamma}(k)\int d^3j[u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+2u_\beta^{>c}(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+2u_\beta^{>c}(\mathbf{j},t)u_\gamma^{>b}(\mathbf{k}-\mathbf{j},t)]\ , \tag{A4}$$

since the substitution of $u^{>b}$ into the corresponding resolvable scale equation will not generate useful information. Thus $u^>$ can be replaced by $u^{>c}$ in the second and last terms of Eq. (A4), in the spirit of keeping just the lowest inhomogeneous effect. One now proceeds analogously to the forced-decay case, noting the statistical independence of $u^<$ and $u^>$ in the second term of Eq. (A4,

APPENDIX B: EFFECTS OF THE SPECTRAL GAP

In this appendix, we examine the effects of a spectral gap in this reformalized RNG and show that, as in the earlier formalism [10,11], triple nonlinearities are not generated if there is a spectral gap. It should be noted it is the existence of a spectral gap that led to the original concept of eddy viscosity in gas dynamics. The spectral gap is a commonly used simplifying assumption in calculating magnetohydrodynamic (MHD) transport coefficients [44–48] and the modeling of interplanetary MHD fluctuations [49,31,32]. Although in the latter case, the small- and large-scale structures are well separated in size [49], numerical results indicate that the spectral gap will be filled in within a few eddy-turnover times. The consequences of invoking the spectral gap have been explored in free-decaying turbulence [10]. Here we consider forced turbulence.

The subgrid-scale equation for the velocity field, in the presence of a spectral gap, now takes the form

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^>(\mathbf{k},t)=f_\alpha^>(\mathbf{k},t)+2M_{\alpha\beta\gamma}(k)\int d^3j\,u_\beta^>(\mathbf{j},t)\times u_\gamma^<(\mathbf{k}-\mathbf{j},t) \tag{B1}$$

where we retain only the relevant terms. The corresponding resolvable scale momentum reduces to

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^<(\mathbf{k},t)=f_\alpha^<(\mathbf{k},t)+M_{\alpha\beta\gamma}(k)\int d^3j[u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+u_\beta^>(\mathbf{j},t)u_\gamma^>(\mathbf{k}-\mathbf{j},t)]\ . \tag{B2}$$

On applying ensemble averaging to Eq. (B2), we obtain

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^<(\mathbf{k},t)=f_\alpha^<(\mathbf{k},t)+M_{\alpha\beta\gamma}(k)\int d^3j[u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+\langle u_\beta^>(\mathbf{j},t)u_\gamma^>(\mathbf{k}-\mathbf{j},t)\rangle]\ . \tag{B3}$$

Using Eq. (31),

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^<(\mathbf{k},t)+\int_{-\infty}^{t}d\tau\,\eta_0(k,t,\tau)u_\alpha^<(\mathbf{k},\tau)=f_\alpha^<(\mathbf{k},t)+M_{\alpha\beta\gamma}(k)\int d^3j\,u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)\ . \tag{B4}$$

The eddy-damping function will not have any triple nonlinearity effects. This also holds for all further RNG iterations.

Thus the spectral gap assumption precludes the existence of the triple nonlinearities and no additional assumptions are needed to achieve closure. However, if there are no spectral gaps then there is no way to avoid the triple nonlinearities.

APPENDIX C: THE RENORMALIZED EDDY VISCOSITY IN FREE-DECAY AND FORCED TURBULENCE

In this appendix, we briefly summarize how the renormalized eddy viscosity is derived in the case of free-decay [10] and forced turbulence [11] and then clarify the numerical scaling used in these earlier calculations.

1. Free-decay turbulence [10]

After the removal of n subgrid shells it has been shown [10] that the eddy-viscosity recursion relation is given by

$$\nu_{n+1}(k)=\nu_n(k)+\delta\nu_n(k)\ , \tag{C1}$$

where

$$\delta\nu_n(k)=2\sum_{i=0}^{n}\int d^3j\frac{L_{kj}Q(|\mathbf{k}-\mathbf{j}|)}{\nu_i(j)j^2k^2}\ , \tag{C2}$$

with the coefficient L_{kj} defined by

$$L_{kj}=-2M_{\alpha\beta\gamma}(\mathbf{k})M_{\beta\beta'\gamma'}(\mathbf{j})D_{\beta'\gamma}(\mathbf{k}-\mathbf{j})D_{\gamma'\alpha}(\mathbf{k}) \tag{C3}$$

and $Q(|\mathbf{k}-\mathbf{j}|)$ is related to the energy spectrum $E(|\mathbf{k}-\mathbf{j}|)$

$$Q(|\mathbf{k}-\mathbf{j}|)=\frac{E(|\mathbf{k}-\mathbf{j}|)}{4\pi|\mathbf{k}-\mathbf{j}|^2}\ . \tag{C4}$$

In free-decaying turbulence, it is assumed that a Kolmogorov energy spectrum is maintained by some means

$$E(k)=C_k\varepsilon^{2/3}k^{-5/3}\ , \tag{C5}$$

where C_k is the Kolmogorov constant and ε is the energy dissipation rate.

The RNG transformation consists in defining

$$k=k_{n+1}\tilde{k}=h^{n+1}k_0\tilde{k}\ , \tag{C6}$$

where $0<h<1$ measures the coarseness of the subgrid partitioning, Eq. (5), and introducing the renormalized eddy viscosity ν_n^* by

$$\nu_n(k_{n+1}\tilde{k})=C_k^{1/2}\varepsilon^{1/3}k_{n+1}^{-4/3}\nu_n^*(\tilde{k})\ , \tag{C7}$$

so that the renormalized recursion relation now takes the form

$$\nu_{n+1}^*(\tilde{k})=h^{4/3}[\nu_n^*(h\tilde{k})+\delta\nu_n^*(h\tilde{k})]\ , \tag{C8}$$

with

$$\delta\nu_n^*(\tilde{k})=\frac{1}{2\pi\tilde{k}^2}\sum_{i=0}^{n}h^{-4i/3}\int d^3\tilde{j}\frac{L_{\tilde{k}\tilde{j}}|\tilde{\mathbf{k}}-\tilde{\mathbf{j}}|^{-11/3}}{\nu_{n-1}^*(h^i\tilde{j})\tilde{j}^2}\ . \tag{C9}$$

The recursion relation, Eq. (C8), is now iterated to find the fixed point ν^* and this is plotted in Fig. 3 as the RNG momentum eddy viscosity. Now from Eqs. (C7) and (C8), we see that the normalized eddy viscosity ν^* is related to the transport eddy viscosity ν by

$$\nu^*(\tilde{k})=\frac{\nu(\tilde{k})}{[E(\tilde{k})/\tilde{k}]^{1/2}}\ . \tag{C10}$$

Thus our RNG fixed point ν^* as calculated in Ref. [10] corresponds to the normalized eddy viscosity defined in typical LES calculations and direct numerical simulations.

2. Forced turbulence [11]

In the case of forcing given by Eq. (13), with $y=3$, the eddy-viscosity recursion relation is given by Eq. (C1) but with

$$\delta\nu_n(k)=\frac{D_0}{k^2}\sum_{i=0}^{n}\int d^3j\frac{L_{kj}|\mathbf{k}-\mathbf{j}|^{-3}}{\nu_i(j)j^2\nu_n(\mathbf{k}-\mathbf{j})|\mathbf{k}-\mathbf{j}|^2}\ . \tag{C1$'$}$$

The extra factor of $\nu_n(\mathbf{k}-\mathbf{j})|\mathbf{k}-\mathbf{j}|^2$ arises because of the presence of this factor in the Green's function [cf. Eq. (8)]. We shall now relate the fixed point of the RNG transformation with the eddy viscosity as defined in LES and direct numerical simulations. The energy spectrum can be determined from the velocity covariance

$$E(k)=2\pi k^2D_{\alpha\beta}(\mathbf{k})\langle u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle\ , \tag{C11}$$

while the velocity covariance is determined from the forcing covariance [11]

$$\langle u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle = \frac{D_0 D_{\alpha\beta}(\mathbf{k})|\mathbf{k}|^{-3}}{2\nu(\mathbf{k})k^2} , \quad \text{(C12)}$$

where $\nu(\mathbf{k})$ is the RNG fixed point, and D_0 defines the intensity of the forcing. From Eqs. (C12) and (C11) we obtain a relationship between the energy spectrum and the RNG eddy viscosity

$$E(\mathbf{k}) = \frac{2\pi D_0 k^{-3}}{\nu(\mathbf{k})} . \quad \text{(C13)}$$

From the RNG scaling transformation, the eddy viscosity $\nu(\mathbf{k})$ can be related to the numerical RNG eddy viscosity $\nu^*(\mathbf{k})$ reported in the figures of Ref. [11]:

$$\nu(\mathbf{k}) = \nu^*(\mathbf{k})(2\pi D_0)^{1/3}k^{-4/3} . \quad \text{(C14)}$$

The energy spectrum can thus be written

$$E(\mathbf{k}) = \frac{(2\pi D_0)^{2/3}k^{-5/3}}{\nu^*(\mathbf{k})} . \quad \text{(C15)}$$

On the other hand, in conventional LES and DNS the normalized eddy viscosity is defined by

$$\nu_{\text{norm}}(\mathbf{k}) \equiv \frac{\nu(\mathbf{k})}{[E(\mathbf{k})/k]^{1/2}} . \quad \text{(C16)}$$

Thus from Eqs. (C13) and (C14) we can relate the numerical RNG eddy viscosity $\nu^*(\mathbf{k})$ reported by us in Ref. [11] to those numerical values that we would have found if we had normalized our RNG eddy viscosity as in LES and DNS.

$$\nu_{\text{norm}}(\mathbf{k}) = [\nu^*(\mathbf{k})]^{3/2} . \quad \text{(C17)}$$

In the figures of this paper, we use the conventional LES normalization and so must apply the relation (C17) to the numerical values given in Ref. [11] for the case of forced turbulence.

[1] K. G. Wilson, Rev. Mod. Phys. **47**, 773 (1975).
[2] K. G. Wilson and J. Kogut, Phys. Rep. **12C**, 75 (1974).
[3] K. G. Wilson, Rev. Mod. Phys. **55**, 583 (1983).
[4] K. G. Wilson, Adv. Math. **16**, 170 (1975).
[5] D. Forster, D. Nelson, and M. Stephen, Phys. Rev. A **16**, 732 (1977).
[6] J. D. Fournier and U. Frisch, Phys. Rev. A **28**, 1000 (1982).
[7] V. Yakhot and S. A. Orszag, J. Sci. Comput. **1**, 3 (1986).
[8] R. H. Kraichnan, Phys. Fluids **30**, 2400 (1987).
[9] H. A. Rose, J. Fluid Mech. **81**, 719 (1977).
[10] Y. Zhou, G. Vahala, and M. Hossain, Phys. Rev. A **37**, 2590 (1988).
[11] Y. Zhou, G. Vahala, and M. Hossain, Phys. Rev. A **40**, 5865 (1989).
[12] Y. Zhou and G. Vahala, Phys. Rev. A **147**, 43 (1990).
[13] L. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, New York, 1982).
[14] R. S. Rogallo and P. Moin, Annu. Rev. Fluid Mech. **16**, 99 (1984).
[15] J. Smagorinsky, Mon. Weather Rev. **91**, 99 (1963).
[16] J. Deardorff, J. Fluid Mech. **41**, 453 (1970).
[17] J. Deardorff, J. Comput. Phys. **7**, 120 (1971).
[18] Y. Zhou, M. Hossain, and G. Vahala, Phys. Lett. A **139**, 330 (1989).
[19] H. A. Rose and P. L. Sulem, J. Phys. (Paris) **39**, 441 (1978).
[20] R. H. Kraichnan, J. Atmos. Sci. **33**, 1521 (1976).
[21] D. C. Leslie and G. L. Quarini, J. Fluid Mech. **91**, 65 (1979).
[22] J. P. Chollet and M. Lesieur, J. Atmos. Sci. **38**, 2747 (1981).
[23] J. A. Domaradzski, R. W. Metcalfe, R. S. Rogallo, and J. J. Riley, Phys. Rev. Lett. **58**, 547 (1987).
[24] M. Lesieur and R. Rogallo, Phys. Fluids A **1**, 718 (1989).
[25] J. R. Chasnov, Ph.D. thesis, Columbia University, 1990.
[26] J. R. Chasnov, Phys. Fluids A **3**, 188 (1991).
[27] Y. Zhou, Phys. Rev. A **43**, 7049 (1991).
[28] G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, New York, 1953).
[29] J. D. Fourier and U. Frisch, Phys. Rev. A **17**, 747 (1978).
[30] D. C. Leslie, *Developments in the Theory of Turbulence* (Oxford University Press, New York, 1974).
[31] Y. Zhou and W. H. Matthaeus, J. Geophys. Res. **95**, 10 291 (1990).
[32] Y. Zhou and W. H. Matthaeus, J. Geophys. Res. **95**, 14 881 (1990).
[33] L. L. Kichatinov, Magnetohydrodynamics **21**, 105 (1985).
[34] Y. Zhou, Phys. Rev. A **41**, 5683 (1990).
[35] H. K. Moffatt, Rep. Prog. Phys. **46**, 621 (1983).
[36] M. J. Lee, Ph.D. thesis, Stanford University, 1985.
[37] A. Leonard, Adv. Geophys. A **18**, 237 (1974).
[38] W. P. Dannevik, V. Yakhot, and S. A. Orszag, Phys. Fluids **30**, 2021 (1987).
[39] S. Panchev, *Random Functions and Turbulence* (Pergamon, New York, 1971).
[40] W. C. Wang, Sol. Phys. **53**, 507 (1977).
[41] D. Montgomery, M. Brown, and W. H. Matthaeus, J. Geophys. Res. **92**, 282 (1987).
[42] R. H. Kraichnan, J. Fluid Mech. **47**, 525 (1971).
[43] M. Hossain, Phys. Lett. A **161**, 277 (1992).
[44] D. Biskamp, Plasma Phys. Contr. Fusion **26**, 311 (1984); D. Biskamp and H. Welter, Phys. Lett. **96A**, 25 (1983).
[45] D. Montgomery and T. Hatori, Plasma Phys. Contr. Fusion **26**, 717 (1984).
[46] D. Montgomery and H. Chen, Plasma Phys. Contr. Fusion **26**, 1199 (1984).
[47] H. Chen and D. Montgomery, Plasma Phys. Contr. Fusion **29**, 205 (1987).
[48] Y. Zhou and W. H. Matthaeus, Geophys. Res. Lett. **16**, 755 (1985).
[49] A. J. Hundhausen, *Coronal Expansion and Solar Wind* (Springer, New York, 1972).